

J. W. Maluf and A. Goya*

Instituto de Física

Universidade de Brasília

C.P. 04385

70.919-970 Brasília, DF

Brazil

Abstract

We consider the class of space-time defects investigated by Puntigam and Soleng. These defects describe space-time dislocations and disclinations (cosmic strings), and are in close correspondence to the actual defects that arise in crystals and metals. It is known that in such materials dislocations and disclinations require a small and large amount of energy, respectively, to be created. The present analysis is carried out in the context of the teleparallel equivalent of general relativity (TEGR). We evaluate the gravitational energy of these space-time defects in the framework of the TEGR and find that there is an analogy between defects in space-time and in continuum material systems: the total gravitational energy of space-time dislocations and disclinations (considered as idealized defects) is zero and infinite, respectively.

PACS numbers: 04.20.Cv, 04.20.Fy, 04.90.+e

(*) e-mail: wadih@fis.unb.br

I. Introduction

Space-time defects are assumed to play an important role in the large-scale structure of the universe. The cosmic strings are prominent examples. The belief is that they may have acted as seeds in the process of formation of galaxies and cluster of galaxies, hence contributing to the problem of the origin of the initial density fluctuations[1]. The two kinds of space-time defects, dislocations and disclinations (cosmic strings), are geometrical constructions that stem from actual defects that occur in crystals and metals[2]. A discussion of the differential geometry and topology of these defects, in the context of condensed matter physics, is given in Ref. [3], and a pictorial exposition in Ref. [4]. The similarity between some gravitational field configurations and physical realizations of material systems like crystals and metals is by itself a very interesting feature.

Topological defects like dislocations and disclinations, which are also called *Volterra distortions*, have recently been considered in the literature in the context of the Einstein-Cartan theory[5]. A systematic investigation of the analogy between distortions of solids and defect structures in Riemann-Cartan manifolds has been carried out in the latter reference. One outcome of this investigation is the classification of dislocations and disclinations in Riemann-Cartan space-times. Since the Einstein-Cartan field equations considered in Ref. [5] have an effective Einsteinian form[5, 6], these Volterra distortions are also defect structures in Einstein's general relativity.

In the framework of general relativity space-time defects can be characterized by metric functions that, at least in the examples considered here, can be transformed into a flat space-time metric away from the defect axis. It is then concluded that the defect is concentrated in the z axis and that it is mathematically realized as delta functions with support in the axis. Such interpretation is given, for instance, in Refs. [5, 7, 8, 9].

The energy-momentum tensors that generate simple space-time dislocations and disclinations have been computed in Refs. [7, 9], in the case where the defect parameters are constants. In the context of Einstein's equations it is concluded that the corresponding energy-momentum tensors contain linear and quadratic terms in the two-dimensional delta function. The quadratic terms are simply discarded, and eventually not considered as anomalies of the theory. The final structure of the energy-momentum tensors is then taken to support the above interpretation regarding the localizability of the defect.

In this paper we will address space-time defects in the context of the teleparallel equivalent of general relativity (TEGR)[10, 11, 12]. The TEGR is an alternative geometrical formulation of Einstein's general relativity. The teleparallel geometry is determined by a set of global orthonormal fields, or tetrad fields $e^a{}_\mu$ [13, 14, 15]. The action integral is determined by a particular combination of quadratic terms in the torsion tensor. A definition for the gravitational energy has been established in the Hamiltonian formulation of the TEGR[16, 17].

We will evaluate the gravitational energy of the space-time defects investigated by Puntigam and Soleng[5], and arrive at an interesting result: the gravitational energy of space-time dislocations vanishes, whereas the gravitational energy per unit length for space-time disclinations is finite, and therefore the total energy is infinite. The close analogy with solid continua is clear, since in crystals and metals dislocations and disclinations require a small and large amount of energy, respectively, to be created. The interpretation of the space-time as a continuum with microstructure has interesting consequences.

The determination of the energy-momentum tensors that generate these defects is an important issue that can be easily investigated in the present mathematical setting, by means of our defect model. The emergence of squares of delta functions has been pointed out earlier[7, 9] in the analysis of the chiral string (a dislocation field). Our procedure

will confirm the appearance of such terms.

Notation: space-time indices μ, ν, \dots and $\text{SO}(3,1)$ indices a, b, \dots run from 0 to 3. Latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field $e^a{}_\mu$ yields the definition of the torsion tensor: $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu$. The flat space-time metric is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-+++)$.

II. The Lagrangian formulation of the TEGR

The Lagrangian formulation of the TEGR is formulated in terms of the tetrad field $e^a{}_\mu$, and is given by a sum of quadratic terms in the torsion tensor $T^a{}_{\mu\nu}$, which is related to the anti-symmetric part of Cartan's connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda} \partial_\mu e_{a\nu}$. The curvature tensor constructed out of the latter vanishes identically. This connection defines a space with teleparallelism, or absolute parallelism[15]. The Lagrangian density is given by

$$L(e) = -k e \Sigma^{abc} T_{abc} - L_M, \quad (1)$$

where $k = \frac{1}{16\pi G}$, G is Newton's constant, $e = \det(e^a{}_\mu)$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$ and

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac} T^b - \eta^{ab} T^c).$$

L_M is the Lagrangian density for matter fields. Tetrads transform space-time into $\text{SO}(3,1)$ indices and vice-versa. The trace of the torsion tensor is defined by $T_b = T^a{}_{ab}$. The tensor Σ^{abc} is defined such that

$$\Sigma^{abc} T_{abc} = \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a.$$

The field equations obtained from (1) read

$$\frac{\delta L}{\delta e^{a\mu}} = e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e \left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd} \right) - \frac{1}{4k} e T_{a\mu} = 0, \quad (2)$$

where $e T_{a\mu} = \delta L_M / \delta e^{a\mu}$. It can be shown by explicit calculations[12] that for $T_{a\mu} = 0$ these equations yield Einstein's equations,

$$\frac{\delta L}{\delta e^{a\mu}} \equiv \frac{1}{2} e \left\{ R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e) \right\} = 0.$$

Therefore the energy-momentum tensor appearing in (2) is strictly equivalent to the corresponding tensor on the right hand side of Einstein's equations. Rewriting equation (2) we have

$$\frac{1}{k} T_{a\mu} = e_{a\mu} T_{bcd} \Sigma^{bcd} + \frac{4}{e} e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - 4 \Sigma^{b\nu}{}_a T_{b\nu\mu}. \quad (3)$$

The equivalence of the present teleparallel theory with Einstein's general relativity must be clarified. First we note that the action integral and field equations (2) remain well defined even when $e^a{}_\mu$ is degenerate. Einstein's equations in terms of tetrad fields also allow degenerate tetrad solutions, which are assumed to induce a topology change of the space-time[19]. However, equivalence of (2) with Einstein's equations in the standard metrical form holds only for *non-degenerate* tetrad configurations. We do not consider the possibility of topology change of the space-time and therefore restrict our attention to non-degenerate tetrad configurations only. We remark that the TEGR is necessarily defined on parallelizable manifolds (in the sense of Ref. [20]), whereas the standard general relativity may be defined also on non-parallelizable manifolds.

The expression for the gravitational energy arises in the Hamiltonian formulation of the TEGR[12]. The energy enclosed by a volume V of the three-dimensional space is

given by[16]

$$E_g = \frac{1}{8\pi} \int_V d^3x \partial_i (e T^i) = \frac{1}{8\pi} \int_S dS_i (e T^i) . \quad (4)$$

All field quantities in (4) are restricted to the three-dimensional spacelike hypersurface; e is now the determinant of the triads $e_{(k)j}$ and T^i is the trace of the torsion tensor: $T^i = g^{ik} T_k = g^{ik} e^{(l)j} T_{(l)jk}$, $T_{(l)jk} = \partial_j e_{(l)k} - \partial_k e_{(l)j}$.

This expression has been thoroughly examined in the literature; it yields the ADM energy[16], and a value strikingly close to the irreducible mass of the Kerr black hole[18]. It has also been applied to the analysis of the gravitational energy of simple space-time defects[21]. However, in order to deal with metric tensors of more intricate space-time defects we need a suitable mathematical description of the defect.

III. The defect model

Idealized space-time defects are represented by constant parameters in the metric tensor. For instance, a space-time dislocation is represented by the metric tensor

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + (dz + \gamma d\phi)^2 ,$$

for constant γ . One can safely take derivatives of the metric components for $r \neq 0$. However, for $r = 0$ one has to be cautious, since the metric tensor as well as the corresponding tetrads are singular at $r = 0$, and the emergence of delta functions with support at the z axis is unavoidable. One way of addressing this problem is to consider the metric tensor in cartesian coordinates[7], making use of distributional identities in the (x, y) plane[22].

In this paper we will investigate space-time defects in cylindrical coordinates, by allowing γ to be a suitable function of the distance $r = \sqrt{x^2 + y^2}$, that circumvents the

above mentioned problem of derivatives in the z axis, and that yields the desired physical features. We establish the defect model by choosing $\gamma(r)$ to be

$$\gamma(r) = \begin{cases} \gamma_0, & \text{if } r > r_0 \\ 0, & \text{if } r \leq r_0 \end{cases}. \quad (5)$$

where γ_0 is a constant measure of the defect. As a consequence, the radial derivative of all space-time defects in this article will be given by Dirac's delta function,

$$\frac{d}{dr}\gamma(r) = \gamma'(r) = \gamma_0\delta(r - r_0). \quad (6)$$

In the following section we will need to evaluate the integral of the product of $\gamma(r)$ with $\gamma'(r)$ (Eqs. (22) and (29)). It can be calculated directly,

$$\int_0^{+\infty} dr \gamma'(r) \gamma(r) = \frac{1}{2} \int_0^{+\infty} dr \frac{d}{dr}[\gamma^2(r)] = \frac{1}{2}[\gamma^2(+\infty) - \gamma^2(0)] = \frac{1}{2}\gamma_0^2, \quad (7)$$

or via integration by parts,

$$\int_0^{+\infty} dr \gamma'(r) \gamma(r) = \int_0^{+\infty} dr \frac{d}{dr}[\gamma^2(r)] - \int_0^{+\infty} dr \gamma(r) \frac{d}{dr}[\gamma(r)],$$

yielding the same result. We note that in order to obtain equation (7) it is not necessary to assume any specific value for $\gamma(r_0)$. The latter quantity is model dependent, and therefore is arbitrarily defined.

For a given arbitrary function $f(r)$ we have

$$\begin{aligned} \int_0^{+\infty} dr f(r) \gamma'(r) \gamma(r) &= \frac{1}{2} \int_0^{+\infty} dr f(r) \frac{d}{dr} \gamma^2(r) \\ &= \frac{1}{2} \int_0^{+\infty} dr \frac{d}{dr} \left(f(r) \gamma^2(r) \right) - \frac{1}{2} \int_0^{+\infty} dr \frac{df(r)}{dr} \gamma^2(r), \end{aligned} \quad (8)$$

from what follows

$$\int_0^{+\infty} dr f(r) \gamma'(r) \gamma(r) = \frac{1}{2} \gamma_0^2 f(r_0) . \quad (9)$$

Again, we note that in order to obtain equation (9) it is not necessary to assume any particular value for $\gamma(r_0)$.

As a final step, we take the limit $r_0 \rightarrow 0$. This procedure allows us to evaluate in a straightforward way several expressions of energy-momentum tensors that have been obtained in the literature by means of alternative methods.

IV. The total gravitational energy of dislocations and disclinations

In real crystals disclinations are highly energetic defects (see, for instance, section 9.2 of Ref. [2]), whereas dislocations require a small amount of energy to be formed. We will conclude that there is a similar situation in the context of space-time defects, by applying expression (4) to the evaluation of the gravitational energy.

We will consider the distorted space-times constructed by Puntigam and Soleng[5]. Out of the ten space-times described by these authors we will take into account only those whose spacelike section is time independent. The six defects described from order 1 through 6 are obtained by means of the Volterra process in \mathbf{R}^3 , and are related to the six degrees of freedom of the proper group of motion of the Euclidean group $SO(3) \otimes T(3)$. As mentioned in Ref. [5], space *and* time supported distortions have no analogy in the theory of elasticity. Therefore we will dispense with orders 8, 9 and 10 of Ref. [5] and consider the metric tensors of order 1 through 6, as displayed in Tables 1 and 2.

According to the procedure outlined in the previous section, the quantities β and γ will be assumed *a priori* as functions of the radial distance r according to equation (5).

Table 1: Space-time dislocations

Order	metric tensor
1	$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + 2\frac{\gamma}{r^2}dx(xdy - ydx) + (\frac{\gamma}{r^2})^2(xdy - ydx)^2$
2	$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + 2\frac{\gamma}{r^2}dy(xdy - ydx) + (\frac{\gamma}{r^2})^2(xdy - ydx)^2$
3	$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + (dz + \gamma d\phi)^2$

Table 2: Space-time disclinations

Order	metric tensor
4	$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 2\frac{\beta}{r^2}(zdy - ydz)(xdy - ydx) + (\frac{\beta}{r^2})^2(y^2 + z^2)(xdy - ydx)^2$
5	$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 2\frac{\beta}{r^2}(zdx - xdz)(xdy - ydx) + (\frac{\beta}{r^2})^2(x^2 + z^2)(xdy - ydx)^2$
6	$ds^2 = -dt^2 + dr^2 + (\beta r)^2 d\phi^2 + dz^2$

The energy expression (4) has been obtained by requiring the time gauge condition, which implies a Hamiltonian formulation with a unique time scale. The tetrad fields below satisfy the time gauge condition. All energy expressions will be evaluated first on a cylindrical surface S of radius $R > r_0$ and height L ; then we will make $R \rightarrow \infty$ and $L \rightarrow \infty$. The prime indicate derivative with respect to the radial distance r .

Order 1

The metric tensor given in Table 1 can be rewritten in cylindrical coordinates as

$$ds^2 = -dt^2 + dr^2 + \sigma^2 d\phi^2 + 2\gamma \cos \phi dr d\phi + dz^2, \quad (10)$$

where $\sigma^2 = r^2 + \gamma^2 - 2\gamma r \sin \phi$. We obtain

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \phi & -r \sin \phi + \gamma & 0 \\ 0 & \sin \phi & r \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (11)$$

from what follows the only non-vanishing torsion component $T_{(1)12} = \gamma'$. Out of the latter quantity we calculate $T^1 = -\frac{1}{e^2}\gamma'(\gamma - r \sin \phi)$, $T^3 = 0$, $e = \det(e^a{}_\mu) = \det(e_{(i)j}) = (r - \gamma \sin \phi)$, and therefore

$$E_g = \frac{1}{8\pi} \int_S dS \left[-\frac{1}{e} \gamma' (\gamma - r \sin \phi) \right], \quad (12)$$

where $dS = d\phi dz$. The integration is only on the cylindrical surface determined by $r = R$ because $T^3 = 0$. We will analyze separately the two terms above.

We consider first $-\frac{1}{e}\gamma'\gamma$. In view of Eqs. (5) and (6) it can be written as

$$-\frac{1}{e}\gamma'\gamma = -\frac{1}{2e} \frac{d}{dr}(\gamma^2) = -\frac{1}{2e} \gamma_0^2 \delta(r - r_0). \quad (13)$$

The integral of equation (15) on a cylindrical surface S of radius $R > r_0$ yields

$$-\frac{1}{8\pi} \int_S d\phi dz \frac{1}{2e} \gamma_0^2 \delta(r - r_0) = -\frac{L}{16\pi} \gamma_0^2 \int_{r=R} d\phi \frac{1}{e} \delta(r - r_0) = 0.$$

Next we consider the term $\frac{1}{e}\gamma' r \sin \phi$. We have

$$\frac{1}{e}\gamma' r \sin \phi = \frac{1}{[r - \gamma(r) \sin \phi]} \gamma_0 \delta(r - r_0) r \sin \phi = \frac{1}{r_0} \gamma_0 \delta(r - r_0) r_0 \sin \phi. \quad (14)$$

The integral on S of equation (14) vanishes, what implies the vanishing of equation (12).

Order 2

In cylindrical coordinates the metric tensor for order 2 reads as

$$ds^2 = -dt^2 + dr^2 + \sigma^2 d\phi^2 + 2\gamma \sin \phi dr d\phi + dz^2, \quad (15)$$

where $\sigma^2 = r^2 + \gamma^2 + 2\gamma r \cos \phi$. Out of the metric tensor above we can construct the set of tetrads

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \phi & -r \sin \phi & 0 \\ 0 & \sin \phi & r \cos \phi + \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

The only non-vanishing tensor component is given by $T_{(2)12} = \gamma'$. It is not difficult to obtain $T^1 = -\frac{1}{e^2}\gamma'(\gamma + r \cos \phi)$, $T^3 = 0$ and $e = (r + \gamma \cos \phi)$. The energy contained within a cylindrical surface S is given by

$$E_g = \frac{1}{8\pi} \int_S dS \left[-\frac{1}{e} \gamma' (\gamma + r \cos \phi) \right], \quad (17)$$

Folowing the same reasoning presented above for order 1 we easily conclude that equation (17) vanishes. We remark that had we assumed from the outset that $\gamma = \gamma_0$ is a constant for any r , both in orders 1 and 2, we would also arrive at a vanishing value for the gravitational energy. The assumption that γ is *a priori* given by equation (5) is important in the analysis of orders 4 and 5 and for obtaining the energy-momentum tensors.

Order 3

The simplest set of tetrad fields that yields the metric tensor of order 3 is given by

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \phi & -r \sin \phi & 0 \\ 0 & \sin \phi & r \cos \phi & 0 \\ 0 & 0 & \gamma(r) & 1 \end{pmatrix}. \quad (18)$$

Again there is only one non-vanishing torsion tensor component, $T_{(3)12} = \gamma'$, that yield vanishing values for all torsion traces T^i . Consequently we arrive at $E_g = 0$.

Order 4

In cylindrical coordinates the metric tensor in Table 2 is rewritten as

$$ds^2 = -dt^2 + dr^2 + \sigma^2 d\phi^2 + dz^2 - 2\beta z \sin \phi dr d\phi + 2\beta r \sin \phi d\phi dz, \quad (19)$$

where $\sigma^2 = e^2 + (\beta r \sin \phi)^2 + (\beta z \sin \phi)^2$, and $e = (r - \beta z \cos \phi)$. The tetrad representation of equation (19) reads

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \phi & -r \sin \phi & 0 \\ 0 & \sin \phi & r \cos \phi - \beta z & 0 \\ 0 & 0 & \beta r \sin \phi & 1 \end{pmatrix}. \quad (20)$$

The non-vanishing torsion tensor components are $T_{(2)12} = -z\beta'$, $T_{(2)23} = \beta$, $T_{(3)12} = (\beta + r\beta') \sin \phi$. We obtain $eT^1 = -\frac{1}{e}\beta'z(\beta z - r \cos \phi)$ and $eT^3 = -\beta \cos \phi + \frac{1}{e}\beta'\beta z r \sin^2 \phi$.

Substituting these quantities in expression (4) we find $E_g = E_{g1} + 2E_{g2}$, where

$$E_{g1} = \frac{1}{8\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \int_0^{2\pi} d\phi \left[-\frac{1}{e}\beta'z(\beta z - r \cos \phi) \right], \quad (21)$$

$$E_{g2} = \frac{1}{8\pi} \int_0^R dr \int_0^{2\pi} d\phi \left[-\beta \cos \phi + \frac{1}{e}\beta'\beta z r \sin^2 \phi \right]. \quad (22)$$

E_{g1} is evaluated at the surface of constant radius $R > r_0$ and height L , and $2E_{g2}$ corresponds to the integrations both at $z = \frac{L}{2}$ and at $-\frac{L}{2}$. Repeating the arguments that led to equations (13) and (14) we conclude that the first and second terms in E_{g1} vanish under integration.

The first term in E_{g2} also vanishes under integration with respect to ϕ . The second term yields a non-vanishing result. By making $z = \frac{L}{2}$ and considering

$$\beta'(r)\beta(r) = \frac{1}{2} \frac{d}{dr}[\beta(r)]^2 = \frac{1}{2} \beta_0^2 \delta(r - r_0) , \quad (23)$$

we find

$$\begin{aligned} E_{g2} &= \frac{1}{8\pi} \frac{L}{2} \int_0^{2\pi} d\phi \sin^2 \phi \int_0^R dr \frac{1}{2} (\beta_0)^2 \delta(r - r_0) \left[\frac{r}{r - \beta(r) \frac{L}{2} \cos \phi} \right] \\ &= \frac{L}{32\pi} (\beta_0)^2 \int_0^{2\pi} d\phi \sin^2 \phi = \frac{L}{32} (\beta_0)^2 . \end{aligned} \quad (24)$$

Note that the use of equation (23) in this calculation is essentially equivalent to using equation (9). Thus the total energy contained within a cylinder of height L is given by

$$E_g = L \left(\frac{\beta_0}{4} \right)^2 . \quad (25)$$

Therefore E_g diverges in the limit $L \rightarrow \infty$.

Order 5

The calculations here are similar to those of order 4. The metric tensor of order 5 in cylindrical coordinates is written as

$$ds^2 = -dt^2 + dr^2 + \sigma^2 d\phi^2 + dz^2 - 2\beta z \cos \phi dr d\phi + 2\beta r \cos \phi d\phi dz , \quad (26)$$

where $\sigma^2 = e^2 + (\beta r \sin \phi)^2 + (\beta z \sin \phi)^2$, and $e = (r + \beta z \sin \phi)$. This space-time disclination can be described by the set of tetrad fields

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \phi & -r \sin \phi - \beta z & 0 \\ 0 & \sin \phi & r \cos \phi & 0 \\ 0 & 0 & \beta r \cos \phi & 1 \end{pmatrix}. \quad (27)$$

Three components of the torsion tensor are non-vanishing: $T_{(1)12} = -z\beta'$, $T_{(1)23} = \beta$, $T_{(3)12} = (\beta + r\beta') \cos \phi$. It is not difficult to obtain $eT^1 = -\frac{1}{e}\beta'z(\beta z + r \sin \phi)$ and $eT^3 = -\beta \sin \phi + \frac{1}{e}\beta'\beta z r \cos^2 \phi$. The total gravitational energy is given by $E_g = E_{g1} + 2E_{g2}$, where

$$E_{g1} = \frac{1}{8\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \int_0^{2\pi} d\phi \left[-\frac{1}{e}\beta'z(\beta z + r \sin \phi) \right], \quad (28)$$

$$E_{g2} = \frac{1}{8\pi} \int_0^R dr \int_0^{2\pi} d\phi \left[-\beta \sin \phi + \frac{1}{e}\beta'\beta z r \cos^2 \phi \right]. \quad (29)$$

E_{g1} is evaluated at the surface of constant radius $R > r_0$, and $2E_{g2}$ corresponds to the integrations at $z = \pm \frac{L}{2}$. Following precisely the same steps that led to expression (24) we find that E_{g1} and the integration of the first term of E_{g2} vanish, and

$$E_g = 2E_{g2} = 2\frac{L}{32\pi}(\beta_0)^2 \int_0^{2\pi} d\phi \cos^2 \phi = L \left(\frac{\beta_0}{4} \right)^2, \quad (30)$$

from what we conclude that the total energy E_g diverges in the limit $L \rightarrow \infty$.

Order 6

The metric tensor for order 6 describes the usual cosmic string. The simplest realization of the tetrad fields that yield the metric tensor in Table 2 is

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\beta \sin \phi & 0 \\ 0 & \sin \phi & \beta \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (31)$$

The determinant is $e = \beta r$ and the two non-vanishing torsion tensor components are $T_{(1)12} = (1 - \beta - r\beta') \sin \phi$, $T_{(2)12} = -(1 - \beta - r\beta') \cos \phi$. We obtain $eT^1 = (1 - \beta - r\beta')$ and $T^3 = 0$. In this order the defect function is defined by

$$\beta(r) = \begin{cases} 1, & \text{if } r \leq r_0 \\ \beta_0, & \text{if } r > r_0 \end{cases},$$

from what follows $\beta'(r) = -(1 - \beta_0)\delta(r - r_0)$. For $r_0 \rightarrow 0$ we obtain

$$E_g = \frac{1}{8\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \int_0^{2\pi} d\phi (1 - \beta - r\beta') = \frac{L}{4} (1 - \beta_0), \quad (32)$$

which is the well-known expression for the energy of the cosmic string. Again the energy diverges in the limit $L \rightarrow \infty$. Therefore for the three disclinations we obtain a finite value for the gravitational energy per unit length of the defect.

V. Energy-momentum tensors

An important issue concerning the physical viability of the space-time defects considered here is the determination of the energy-momentum tensors that generate the defects. We anticipated in section I that in previous investigations of some of these defects (orders 3 and 6) there appeared linear *and* quadratic terms in the delta function. In this section we will present the energy-momentum tensors corresponding to orders 1 through 6, by means of expression (3), and will also arrive at expressions containing squares of the

delta function. We remark that it is possible to demonstrate by explicit calculations that expression (3) is symmetric, i.e., $T_{\mu\nu} = e_a{}^\mu T_{a\mu} = T_{\nu\mu}$.

The calculations are quite lengthy, but otherwise straightforward. We need to evaluate all components $T_{\lambda\mu\nu} = e^a{}_\lambda T_{a\mu\nu}$ and of the tensor $\Sigma^{\mu\nu\lambda}$ defined in section II. All components of $T_{a\mu\nu}$ were evaluated in the previous section. In what follows we will just present the final expressions of the non-vanishing components of $T_{\mu\nu}$ exactly as they arise from the calculations (except for orders 4 and 5; see below). The expressions will be given in terms of the functions $\beta(r)$, $\gamma(r)$ and their derivatives. Therefore it will be understood that $\beta'(r) = \beta_0\delta(r - r_0)$ (for order 6, $\beta'(r) = -(1 - \beta_0)\delta(r - r_0)$) and $\gamma'(r) = \gamma_0\delta(r - r_0)$, for arbitrarily small r_0 .

Order 1

$$\frac{T_{00}}{k} = -\frac{T_{33}}{k} = -\frac{2}{e}(\gamma - r \sin \phi) \partial_r \left[\frac{\gamma'}{e} \right] - \frac{2}{e^2}(\gamma')^2 + \frac{2}{e} \gamma \gamma' \cos^2 \phi .$$

Order 2

$$\frac{T_{00}}{k} = -\frac{T_{33}}{k} = -\frac{2}{e}(\gamma + r \cos \phi) \partial_r \left[\frac{\gamma'}{e} \right] - \frac{2}{e^2}(\gamma')^2 - \frac{2}{e} \gamma \gamma' \sin^2 \phi .$$

Order 3

$$\frac{T_{00}}{k} = -\frac{T_{11}}{k} = -\frac{(\gamma')^2}{2e^2} , \quad \frac{T_{22}}{k} = -2\gamma e \partial_r \left[\frac{\gamma'}{e} \right] - \frac{(\gamma')^2}{2} + \gamma^2 \left[\frac{3(\gamma')^2}{2e^2} \right] ,$$

$$\frac{T_{23}}{k} = -e \partial_r \left[\frac{\gamma'}{e} \right] + \gamma \left[\frac{3(\gamma')^2}{2e^2} \right] , \quad \frac{T_{33}}{k} = \frac{3(\gamma')^2}{2e^2} .$$

Order 4

$$\frac{T_{00}}{k} = -\frac{2}{e}z(\beta z - r \cos \phi) \partial_r \left[\frac{\beta'}{e} \right] + \frac{2}{e^3}\beta\beta'(z^2 + r^2) \sin^2 \phi ,$$

$$\frac{T_{11}}{k} = -\frac{2}{e^2}\beta\beta'r \sin^2 \phi ,$$

$$\frac{T_{12}}{k} = \frac{1}{e}\beta^2 z r^2 \sin^3 \phi \partial_r \left[\frac{\beta'}{e} \right] + \frac{1}{e}\beta\beta'r \sin \phi \cos \phi \left(1 + \frac{r^2}{e^2} \right) ,$$

$$\frac{T_{13}}{k} = \frac{1}{e}\beta z r \sin^2 \phi \partial_r \left[\frac{\beta'}{e} \right] + \frac{1}{e^2}\beta'(3e \cos \phi - \beta z) - \frac{2}{e^2}\beta\beta'z \cos^2 \phi - \frac{1}{e^3}\beta\beta'z r \sin^2 \phi ,$$

$$\frac{T_{22}}{k} = -2\beta r^3 \sin^2 \phi \partial_r \left[\frac{\beta'}{e} \right] - \frac{6}{e^2}\beta\beta'r^3 \sin^2 \phi ,$$

$$\frac{T_{23}}{k} = \frac{1}{e}r \sin \phi (\beta^2 z^2 - r^2) \partial_r \left[\frac{\beta'}{e} \right] - \frac{2}{e}\beta\beta'z \sin \phi \cos \phi + \frac{3}{e^2}\beta' \sin \phi (\beta z r \cos \phi - r^2) ,$$

$$\frac{T_{33}}{k} = \frac{2}{e}z(\beta z - r \cos \phi) \partial_r \left[\frac{\beta'}{e} \right] - \frac{2}{e^3}\beta\beta'z^2 \sin^2 \phi .$$

Order 5

$$\frac{T_{00}}{k} = -\frac{2}{e}z(\beta z + r \sin \phi) \partial_r \left[\frac{\beta'}{e} \right] + \frac{2}{e^3}\beta\beta'(z^2 + r^2) \cos^2 \phi ,$$

$$\frac{T_{11}}{k} = -\frac{2}{e^2}\beta\beta'r \cos^2 \phi ,$$

$$\frac{T_{12}}{k} = \frac{1}{e}\beta^2 z r^2 \cos^3 \phi \partial_r \left[\frac{\beta'}{e} \right] - \frac{1}{e}\beta\beta'r \sin \phi \cos \phi \left(1 + \frac{r^2}{e^2} \right) ,$$

$$\frac{T_{13}}{k} = \frac{1}{e}\beta z r \cos^2 \phi \partial_r \left[\frac{\beta'}{e} \right] - \frac{1}{e^2}\beta'(3e \sin \phi + \beta z) + \frac{2}{e^2}\beta\beta' z \sin^2 \phi - \frac{1}{e^3}\beta\beta' z r \cos^2 \phi ,$$

$$\frac{T_{22}}{k} = -2\beta r^3 \cos^2 \phi \partial_r \left[\frac{\beta'}{e} \right] - \frac{6}{e^2}\beta\beta' r^3 \cos^2 \phi ,$$

$$\frac{T_{23}}{k} = \frac{1}{e}r \cos \phi (\beta^2 z^2 - r^2) \partial_r \left[\frac{\beta'}{e} \right] + \frac{2}{e}\beta\beta' z \sin \phi \cos \phi - \frac{3}{e^2}\beta' \cos \phi (\beta z r \sin \phi - r^2) ,$$

$$\frac{T_{33}}{k} = \frac{2}{e}z(\beta z + r \sin \phi) \partial_r \left[\frac{\beta'}{e} \right] - \frac{2}{e^3}\beta\beta' z^2 \cos^2 \phi ,$$

Order 6

$$\frac{T_{00}}{k} = -\frac{T_{33}}{k} = \frac{2}{e}\partial_r \left[1 - \beta - r\beta' \right] .$$

For each order e is the determinant obtained in the previous section.

The full expressions for the $T_{\mu\nu}$ components of orders 4 and 5 are very intricate. We have just presented the terms that are linear in β' . These are the relevant terms to be considered, according to the suggestion pointed out in Refs. [7, 9]. In these latter references it is argued that the terms quadratic in the delta function are devoid of physical meaning, and that they should be ignored.

The terms *linear* in the defects in the expressions above for orders 3 and 6 are in total agreement with the corresponding expressions obtained in Refs. [7, 9], except that in these references the calculations were carried out in cartesian coordinates (the quadratic terms were not presented in [7, 9]).

VI. Discussion

The results of section IV indicate that there is an analogy between space-time defects and defects in continuum material systems. In the latter, dislocations and disclinations are low and high energy defects, respectively. For the idealized defects considered in this work we obtained an analogous result.

Deformations of metals are explained entirely in terms of dislocations. In the analysis of the plastic properties of metals it is made no reference to disclinations[23]. Deformation of such materials is due to the mobility and multiplication of dislocations. However, deformations of the space-time that would give rise to the formation of matter structure in the universe are presently described by disclinations similar to order 6. The value of the energy per unit length (32) for the usual cosmic string is well-known from previous studies[1]. To our knowledge, energies (25) and (30) have not been calculated so far. As for the dislocations, all have vanishing total energy.

The teleparallel geometry is determined by the choice of a global orthonormal set of tetrad fields. It is established by just declaring a particular orthonormal frame to be the set of tetrad fields for the space-time[24]. All tetrad fields considered here yield vanishing torsion tensor by requiring the physical parameters β_0 and γ_0 to vanish. Alternatively, if we require the latter parameters to vanish, then in cartesian coordinates all tetrad fields reduce to

$$e^a{}_{\mu}(t, x, y, z) = \delta^a_{\mu} . \quad (33)$$

It is legitimate to ask whether different choices for $e_{a\mu}$, in the six cases above, would render different results for the energy. The answer is that the *total* energy does not depend on the particular choice of an orthonormal frame, provided the tetrad fields are related

to each other via a local $SO(3,1)$ transformation such that the transformation matrices satisfy appropriate boundary conditions (section VI of Ref. [25]). The transformed tetrad fields must also reduce to the form given by equation (33) for vanishing physical parameters. In order to establish the above mentioned analogy it suffices to evaluate the total gravitational energy of the space-time.

Expression (4) for the gravitational energy was obtained from the Hamiltonian formulation where the time gauge condition for the tetrad field ($e_{(i)}^0 = 0$) was imposed from the outset. If we consider the Hamiltonian formulation of the TEGR without fixing the time gauge condition, there arises a total divergence in the Hamiltonian constraint that plays a role similar to expression (4), and that is identified with the energy-momentum of the gravitational field[26]. However, it is possible to prove[27] that by requiring $e_{a\mu}$ to satisfy (*a posteriori*) the time gauge condition the latter expression for the gravitational energy exactly coincides with expression (4).

Acknowledgements

A. G. is supported by CAPES, Brazil.

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